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Some Notes on Boundaries with Application

Siham J. Al - Sayyad

Abstract

We established some certain boundary sets in topological spaces.

We obtain the following results :

(i) $\partial(A \cup B) = \partial A \cup \partial B$ *iff* $\partial(A \cap B) \cup (\partial A \cap \partial B) \subseteq \partial(A \cup B)$

(ii) $\partial\partial(A \cup B) \subseteq \partial A \cup \partial\partial B$

(iii) $\partial A \cap \partial B \subseteq \partial(A \cup B)$ *implies* $\partial\partial(A \cap B) \subseteq \partial\partial A \cup \partial\partial B$

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§ 1. Introduction

We shall establish some properties of boundary sets in topological spaces. X denotes the fixed topological space with no specific property or separation axiom. $A^\circ = \text{int} A, \overline{A} = clA$ are the interior and the closure of the subset A in X as usually. The boundary of A is $\partial A = \overline{A} \cap \overline{X} = \overline{A} - A^\circ$. The boundary of ∂A will be written $\partial \partial A$. It is well known that if A is open or closed then $\partial \partial A = \partial A$ holds, i.e ∂A is nowhere dense.

A more general statement also be proved: Nowhere dense subsets are precisely the subsets of those special boundary sets with form $\partial(A \cup \partial B)$ where A is semi-open or semi-closed after remembering the basic inclusion $\partial(A \cup \partial B) \subseteq \partial A \cup \partial\partial B$.

Recall that a subset A is called semi-open (resp. semi-closed) iff there exists an open G (resp. closed K) with $G \subseteq A \subseteq \overline{G}$ $(resp. k^\circ \subseteq A \subseteq K)$, see [1-4]. Hence boundaries of semi-open or semi-closed subsets are also nowhere dense since $\partial A \subseteq \partial G$ or $\partial A \subseteq \partial K$ hold respectively. In fact boundaries of semi-open (resp. semi-closed) sets are precisely boundaries

of open sets since $\partial A = \partial\partial A = \partial(X - \partial A)$ hold if A is semi-open or semi-closed. Notice also that in any space with a dense subset D with empty interior, closed subsets are nothing but boundaries since $\partial((k^{\circ} \cap D) \cup \partial K) = K$ holds for any closed K in such spaces.
Thus closed subsets are boundaries in Euclidean spaces, see [5].

The following basic facts will be used frequently:

$G \subseteq X$ is open iff $cl(G \cap \overline{A}) = cl(G \cap A)$ for all $A \subseteq X$ (1.1)

$K \subseteq X$ is closed iff $\text{int}\,(K \cup A^{\circ}) = \text{int}\,(K \cup A)$ for all $A \subseteq X$ (1.2)

$$A^{\circ} \cap \overline{B} \subseteq cl(A^{\circ} \cap B), \text{int}(A \cup B) \subseteq A^{\circ} \cup \overline{B} \quad (1.3)$$

$$A \subseteq \partial B \text{ iff } A \subseteq \overline{B} \text{ and } A \cap B^{\circ} = \phi \quad (1.4)$$

$$\partial(A \cap B) \cup \partial(A \cup B) = \partial A \cup \partial B \quad (1.5)$$

§ 2. Main Results

Any result at the sequel stated without any additional condition or hypothesis is true all subsets of X.

Theorem 2.1 : We have the inclusions:

(i) $(\partial A - B) \cup (\partial B - A) \subseteq \partial(A \cup B) \subseteq (\partial A - \overline{B}) \cup (\partial B - \overline{A}) \cup (\partial A \cap \partial B)$,

(ii) $(\partial A \cup B^{\circ}) \cup (\partial B \cap A^{\circ}) \subseteq \partial(A \cap B) \subseteq (\partial A \cap B^{\circ}) \cup (\partial B \cap A^{\circ}) \cup (\partial A \cap \partial B)$,

(iii) $\partial(A \cup \overline{B}) \cup \partial(B \cup \overline{A}) \subseteq \partial(A \cup B) \subseteq \partial(A \cup \overline{B}) \cup \partial(B \cup \overline{A}) \cup (\partial A \cap \partial B)$,

(iv) $\partial(A \cap B^{\circ}) \cup \partial(B \cap A^{\circ}) \subseteq \partial(A \cap B) \subseteq \partial(A \cap B^{\circ}) \cup \partial(B \cap A^{\circ}) \cup (\partial A \cap \partial B)$.

Proof : The proof is trivial. The sight side if (ii) is a consequence of the following

$$\partial(A \cap B) \subseteq (\partial A \cap \overline{B}) \cup (\partial B \cap \overline{A})$$

which is well known or easy to obtain and its left side follows by

$$\partial A \cap B^{\circ} \subseteq (B^{\circ} \cap \overline{A}) - \text{ int } (A \cap B).$$

The left side of (iv) is straight forward after (1.5) and its right side could be obtained by using firstly the right and then the left side of (ii). (i) and (iii) follow respectively by (ii) and (iv). Notice that the unions of (i) and (ii) are mutually disjoint.This completes the proof.

Remark 1 : The following also follows from (ii).

$$\partial A - \overline{B} \subseteq \partial(A - \overline{B}) \subseteq \partial(A - B).$$

Theorem 2.2 : we have:

(i) $\partial(\partial A \cap \partial B) = (\partial\partial A \cap \partial B) \cup (\partial A \cap \partial\partial B)$,

(ii) $\partial(\partial A \cup \partial B) \subseteq (\partial\partial A - \operatorname{int}\partial B) \cup (\partial\partial B - \operatorname{int}\partial A)$,

(iii) $\partial(\partial A \cup \partial B) \subseteq \partial(A \cup B) \cup (\overline{A} \cup \partial B) \cup (\overline{B} \cap \partial A)$,

(iv) $\partial(\partial A \cup \partial B) \subseteq \partial(A \cap B) \cup (\partial A - B^{\circ}) \cup (\partial B - A^{\circ})$.

Proof : We notice that the inclusion

$$\partial(\partial A \cap \partial B) \subseteq (\partial\partial A \cap \partial B) \cup (\partial\partial B \cap \partial A)$$

has already been stated in the proof of Theorem 2.1. The reverse inclusion follows easily by (1.4). (ii) is obtained by the right side of (i) of Theorem 2.1, after noticing

$$\partial\partial A \cap \partial\partial B \subseteq \partial\partial A - \operatorname{int}\partial B.$$

Also Notice that , one gets the following by (i) of Theorem 2.1,

$$\partial(\partial A \cup \partial B) - (\overline{A} \cup \overline{B}) = ((\partial A \cup \partial B - \overline{A}) \cup ((\partial A \cup \partial B) - \overline{B}) \subseteq (\partial B - \overline{A}) \cup (\partial A - \overline{B}) \subseteq \partial(A \cup B).$$

Hence, yielding (iii) is not difficult. (iv) follows directly from (iii). This completes the proof.

Lemma 2.3 :

If A (or B) is open or closed, then

$$\partial(\partial A \cup \partial B) = (\partial\partial A - \operatorname{int}\partial B) \cup (\partial\partial B - \operatorname{int}\partial A).$$

Proof : let A be an open or closed, Then int $\partial A = \phi$ and therefore the intersection of the right side of (ii) of Theorem 2.2 with int $(\partial A \cup \partial B)$ is equal to

$$(\partial\partial B \cup (\partial\partial A - \operatorname{int}\partial B)) \cap \operatorname{int}(\operatorname{int}\partial A \cap \partial B) = \phi$$

and so this inclusion becomes and equality of (1.4).

Corollary 2.1 :

$$\partial(\partial\partial A \cup \partial\partial B) = \partial\partial A \cup \partial\partial B \supseteq \partial(\partial A \cup \partial B),$$

$$\partial(\partial\partial A \cap \partial\partial B) = \partial\partial A \cap \partial\partial B \subseteq \partial(\partial A \cap \partial B),$$

$$\partial(\partial A \cup \partial B) = \partial A \cup \partial B \Rightarrow \partial(\partial A \cap \partial B) = \partial A \cap \partial B.$$

Theorem 2.4 : We have:

(i) $\partial\overline{A} \cup \partial\overline{B} = \partial(\overline{A} \cap \overline{B}) \cup \partial(\overline{A} \cup \overline{B})$,

(ii) $\partial A^\circ \cup \partial B^\circ = \partial(A^\circ \cap B^\circ) \cup \partial(A^\circ \cup B^\circ)$,

(iii) $\partial \overline{A} \cup \partial B^\circ = \partial(\overline{A} - B^\circ) \cup \partial(B^\circ - \overline{A})$.

Proof : Note that the inclusions :

$$(\partial \overline{A} \cup \partial \overline{B}) \cap (\overline{A} \cap \overline{B}) \subseteq \partial(\overline{A} \cap \overline{B})$$

$$(\partial \overline{A} \cup \partial \overline{B}) - (\overline{A} \cap \overline{B}) \subseteq \partial(\overline{A} \cup \overline{B})$$

are derived respectively by (1.4) and the left side of (i) of Theorem 2.1. Therefore (i) follows. The others are consequences of (i). This completes the proof.

Remark 2 : If A and B are both open or closed, then the basic formula (1.5) becomes equality by Theorem 2.4.

Theorem 2.5 : We have the expansions:

(i) $\partial(A \cap B) = \overline{\partial A \cap B^\circ} \cup A \cap \partial_2 B \cup \partial(A^\circ \cap B^\circ) \rightarrow$

(ii) $\partial(A \cup B) = \overline{\partial A - \overline{B}} \cup \overline{A \cap \partial_1 B} \cup \partial(\overline{A} \cup \overline{B})$

Proof : Note that the disjoint subsets

$$\partial_1 A = \overline{A} - A \quad \text{and} \quad \partial_2 A = A - A^\circ$$

have both empty interiors. They satisfy

$\partial A = \partial_1 A \cup \partial_2 A$ for all $A \subseteq X$ and additionally the following equivalencies are clear:

A is open iff $\partial_2 A = \phi$ iff $\partial_1 A = \partial A$

A is closed iff $\partial_1 A = \phi$ iff $\partial_2 A = \partial A$

Furthermore:

$$\begin{aligned}\partial(A \cap B) - \overline{A \cap \partial_2 B} &\subseteq cl((A \cap B) - \overline{\overline{A} \cap \partial_2 B}) \\ &\subseteq cl((A \cap B) - ((A \cap B) - B^\circ)) \\ &= cl(A \cap B^\circ)\end{aligned}$$

Therefore one could get

$$\partial(A \cap B) = \partial(A \cap B^\circ) \cup \overline{A \cap \partial_2 B}$$

Also we note that

$$\begin{aligned}&\partial(A \cap B^\circ) = cl(\overline{A} \cap B^\circ) - \text{int}(A \cap B^\circ) \\ &= (cl(A^\circ \cap B^\circ) \overline{\cup \partial A \cap B}^\circ) - \text{int}(A \cap B^\circ) \\ &= \partial(A^\circ \cap B^\circ) \cup \overline{\partial A \cap B^\circ}\end{aligned}$$

since $cl(\partial A \cap B^\circ)$ is disjoint with int A. Also we note that

$\partial_2(X - A) = \partial_1 A, \partial_2(X - A) = \partial_1 A$

Hence both of the expansions with respect to second set are now established. This completes the proof.

Theorem 2.6 : The following are equivalent:

(i) $\overline{A} \cap \overline{B} = \overline{A \cap B}$,
(ii) $\partial A \cap \partial B \subseteq \partial(A \cap B)$,
(iii) $\partial(A \cap B) = (\overline{A} \cap \partial B) \cup (\overline{B} \cap \partial A)$,
(iv) $\partial(A \cap B) = \partial(A \cap B^{\circ}) \cup \partial(A^{\circ} \cap B) \cup (\partial A \cap \partial B)$.

Proof : The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are straightforward. After noticing $A^{\circ} \cap \partial B \subseteq \partial(A^{\circ} \cap B)$ by (ii) of Theorem 2.1. The condition (iv) evidently implies.

$$\overline{A} \cap \overline{B} = (A^{\circ} \cap B^{\circ}) \cup (A^{\circ} \cap \partial B) \cup (B^{\circ} \cap \partial A) \cup (\partial A \cap \partial B) \subseteq \overline{A \cap B}$$

and therefore (i) is obtained. This completes the proof.

Using Theorem 2.6 we can easily prove the following

Corollary 2.2 : The following are equivalent:
(i) $\text{int}(A \cup B) = A^{\circ} \cup B^{\circ}$,
(ii) $\partial A \cap \partial B \subseteq \partial(A \cup B)$,
(iii) $\partial(A \cup B) = (\partial A - B^{\circ}) \cup (\partial B - A^{\circ})$,
(iv) $\partial(A \cup B) = \partial(A \cup B) = \partial(A \cup \overline{B}) \cup \partial(B \cup \overline{A}) \cup (\partial A \cap \partial B)$.

Theorem 2.7 : The following are equivalent:
(i) $\partial(A \cup B) = \partial A \cup \partial B$,
(ii) $(\partial A \cap \overline{B}) \cup (\partial B \cap \overline{A}) \subseteq \partial(A \cup B)$,
(iii) $\text{int}(A \cap B) = A^{\circ} \cup B^{\circ}, (\partial A \cap B^{\circ}) \cup (\partial B \cap A^{\circ}) \subseteq \partial(A \cup B)$.

Proof : The conditions (iii) and (1.4) imply $\partial A \subseteq \partial(A \cup B) \text{since} (\partial A - B^{\circ})$ and int $(A \cup B)$ are disjoint by (iii).

Corollary 2.3 : $\partial(A \cap B) \cup (\partial A \cap \partial B) \subseteq \partial(A \cup B)$ *iff* $\partial(A \cup B) = \partial A \cup \partial B$.

Proof : The proof following by (ii) of Theorem 2.1 and corollary 2.2 and Theorem 2.7.

Corollary 2.4 : $\partial(A \cap B) = \partial A \cap \partial B$ and
int $(A \cup B) = A^{\circ} \cup B^{\circ}$ imply $\partial(A \cup B) = \partial A \cup \partial B$

Proof : The proof is trivial.

Remark 3 : The equality $\partial(A \bigcup B) = \partial A \bigcup \partial B$ dose not imply $\partial(A \bigcap B) = \partial A \bigcap \partial B$. Just take the triadic relations in [0,1] as A and all the triadic relations in the same interval as B for a counter example in R' .

Theorem 2.8 : We have

(i) $\partial(A \bigcap B) = \partial A \bigcap \partial B$ iff $\overline{A} \bigcap \overline{B} = \overline{A \bigcap B}$ and $\partial(A \bigcap B) \bigcap A^\circ = \partial(A \bigcap B) \bigcap B^\circ$

(ii) $\partial(\partial A \bigcap \partial B) = \partial\partial A \bigcap \partial\partial B$ iff $\partial A \bigcap \partial B = \operatorname{int}\partial B \bigcap \partial A$.

Proof : (i) is clear

(ii) Noting that

$\partial(\partial A \bigcap \partial B) \subseteq \partial\partial A$ *iff* $\operatorname{int}\partial A \bigcap \partial B \subseteq \operatorname{int}(\partial A \bigcap \partial B)$

by (1.4) and applying corollary 2.1, and the result follows.

Theorem 2.9 : The following are equivalent:

(i) $\partial(\partial A \bigcup \partial B) = \partial\partial A \bigcup \partial\partial B$,

(ii) $\partial A \bigcap \operatorname{int}(\partial A \bigcup \partial B) = \operatorname{int}\partial A$ and
$\partial B \bigcap \operatorname{int}(\partial A \bigcup \partial B) = \operatorname{int}\partial B$.

(iii) $\partial A \bigcap \partial B \bigcap \operatorname{int}(\partial A \bigcup \partial B) = \operatorname{int}(\partial A \bigcap \partial B)$,

(iv) $\partial(\partial A \bigcap \partial B) = \partial\partial A \bigcap \partial\partial B$ *and* $\operatorname{int}(\partial A \bigcup \partial B) = \operatorname{int}\partial A \bigcup \operatorname{int}\partial B$,

(v) $\partial(\partial A \bigcap \partial B) = \partial\partial A \bigcap \partial\partial B \bigcap \partial(\partial A \bigcup \partial B)$.

Proof : (i) $\Leftrightarrow$ (ii): Note that the condition (i) implies

$$\partial A \bigcap \operatorname{int}(\partial A \bigcup \partial B) \subseteq \operatorname{int}\partial A \text{ and its dual one.}$$

(ii) $\Leftrightarrow$ (iii): Necessity is clear. For the proof of sufficiency we note that one gets the following by using condition (iii).

$$\partial A \bigcap \operatorname{int}(\partial A \bigcup \partial B) \subseteq \operatorname{int}(\partial A \bigcap \partial B) \bigcup ((\partial A - \partial B) \bigcap \operatorname{int}(\partial A \bigcup \partial B))$$
$$\subseteq \operatorname{int}\partial A \bigcup ((\operatorname{int}\partial A \bigcup \partial B) - \partial B) \subseteq \operatorname{int}\partial A.$$

(iii) $\Leftrightarrow$ (iv): We note that the condition (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) implies

$$\partial\partial A \bigcap \partial\partial B \subseteq \partial(\partial A \bigcup \partial B),$$
$$\operatorname{int}\partial A \bigcap \partial B = \operatorname{int}\partial B \bigcap \partial A.$$

So the required implication follows by corollary 2.2 and Theorem 2.8 – ii
(iv)$\Rightarrow$(v): clear by Corollary 2.2, since $\partial\partial A\,\partial\partial B \subseteq \partial(\partial A \cup \partial B)$ holds by (iv).
(v)$\Rightarrow$ (i): By using (v) at the last inclusion the following:

$$\partial\partial A \cap \partial\partial B \cap \operatorname{int}(\partial A \cup \partial B) \subseteq \partial(\partial A \cap \partial B) \cap \operatorname{int}(\partial A \cup \partial B)$$
$$\subseteq \partial(\partial A \cup \partial B) \cap \operatorname{int}(\partial A \cup \partial B) = \phi,$$

one yields $\begin{array}{l}\partial\partial A \cap \partial\partial B \subseteq \partial(\partial A \cup \partial B), i.e \\ \operatorname{int}(\partial A \cup \partial B) = \operatorname{int}\partial A \cup \operatorname{int}\partial B.\end{array}$

So all the sufficient conditions of Corollary 2.3 for being equality written (i) hold are now satisfied after (v) and Corollary 2.1.

Remark 4 : Note the difference of Corollary 2.3 with the equivalency of the conditions (i) and (ii) of Theorem 2.9.

Corollary 2.5 : $\partial A \cap \partial B = \phi$ implies

$$\overline{A} \cap \overline{B} = \overline{A \cap B},$$
$$\operatorname{int}(A \cup B) = A^{\circ} \cup B^{\circ},$$
$$\partial(\partial A \cup \partial B) = \partial\partial A \cup \partial\partial B,$$
$$\partial(A \cup B) \cup \partial(A \cap B) = \partial A \cup \partial B.$$

Proof : The first three equalities are easy to prove and so is omitted. Now not that
$(\partial A \cup \partial B) - (\partial(A \cup B) \cup \partial(A \cap B)) = (\partial A \cup \partial B) \cap (\operatorname{int}(A \cup B) - \overline{A \cap B})$
always holds. The hypothesis makes the right side
$(\partial A \cup \partial B) \cap (A^{\circ} - \overline{B}) \cup (B^{\circ} - \overline{A}) = \phi$

Hence (1.5) gives the last equality

Theorem 2.10 : $\partial\partial A = \partial A^{\circ} \cup \partial\overline{A}$

Proof : $\partial\partial A = \partial A - \operatorname{int}\partial A$
$= (\partial A \cap clA^{\circ}) \cup (\partial A \cap cl(X - \overline{A}))$
$= \partial A^{\circ} \cup \partial\overline{A}.$

Corollary 2.6 : If A is open or closed then $\partial\partial A = \partial A$

Proof: This well known result is a direct and easy consequence of Theorem 2.10. Let A be open. Then $\partial\partial A = \partial A \bigcup \partial\overline{A} = \partial A$, since $\partial\overline{A} \subseteq \partial A$ and $\partial A^{\circ} \subseteq \partial A$ are derived by the same Theorem 2.10. This completes the proof.

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Siham J. Al - Sayyad

Department of Mathematics
Faculty of Science
King Abdul - Aziz University
P. O. Box 30305
Jeddah 21477, Saudi Arabia