J.K.A.U. Sci., vol. 1, pp. 231-239 (1409 A.H. / 1989 A.D.

Queues with Multiple Poisson Inputs and Erlang Service Times

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ABSTRACT. The paper deals with the steady-state solution of the queueing system in which (i) different types of units arrive according to independent Poisson distributions with different parameters, (ii) the units are served in order of their arrival and (iii) the service time distributions are Erlang with different parameters. Assuming that all the units arriving at the system wait until they are served, we derived the probability generating functions and the measures of effectiveness of the system.

1. Introduction

An analysis of a single-server queueing system for m different types of units having independent Poisson arrivals with rates λ_i, $i = 1,2,\ldots, m$, and exponential service times with rates μ_i, $i = 1,2, \ldots, m$, has been studied by Ancker and Gafarian[1]. They used the method of successive substitution and derived a recursion relation for the steady-state probability of n units in the queue and a recursion relation for the steady-state probability that some member of a particular type is in service and that n units of any type are in the queue.

Here, a generalization of the above system is studied. Consider a service facility with a single server at which m different types of units arrive singly demanding service. The units are having independent Poisson arrival distributions with parameters λ_i, $i = 1,2, \ldots, m$. All the arriving units join the system and stay there until they complete their services.

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The units are served one at a time and in order of their arrival. The service-time of each type is random and independent of the others and a unit of type-i goes through k_i phases to complete its service. The lengths of time to complete the different phases of service of a unit of type-i are i.i.d. random variables all with distribution:

$$k_i \mu_i \, e^{-k_i \mu_i x}$$

that is the service-time distribution for type-i unit is

$$b_i(x) = \frac{(k_i\mu_i)\,(k_i\mu_i x)^{k_i - 1}\, e^{-k_i\mu_i x}}{(k_i - 1)!}$$

which is Erlang distribution.

Ancker and Gafarian[1] proved that the overall inter-arrival time distribution is negative exponential with parameter $\lambda = \sum_{i=1}^{m} \lambda_i$.

They also obtained the over-all service-time distribution as

$$b(x) = \sum_{i=1}^{m} \frac{\lambda_i}{\lambda} b_i(x)$$

$$\sum_{i=1}^{m} \frac{\lambda_i}{\lambda} \frac{(k_i\mu_i)\,(k_i\mu_i x)^{k_i - 1}\, e^{k_i\mu_i x}}{(k_i - 1)!}$$

The first two moments of this distribution are:

$$E\,(X) = \sum_{i=1}^{m} \frac{\lambda_i}{\lambda} \frac{1}{\mu_i}$$

$$\text{and } E\,(X^2) = \sum_{i=1}^{m} \frac{\lambda_i\,(K_i + 1)}{\lambda\, k_i\, \mu_i^2}, \qquad (1.4)$$

The above system will be studied in the steady-state case, assuming that

$$\lambda \sum_{i=1}^{m} \frac{\lambda_i}{\lambda} \cdot \frac{1}{\mu_i} = \sum_{i-1}^{m} \frac{\lambda_i}{\mu_i} < . \; 1$$

2. The Steady-State Equations and Their Solution

Let P_0 be the probability that the system is empty, and

${}_jP_{ns}$ be the probability that there are n, ($n \geq 1$), units in the system and the unit being served is of type-j and is in the *sth* phase of its service.

For convenience, the phases in each one of the branches of the service facility will be labeled in reverse order. The unit joining the *jth* branch starts its service with the

$k_j th$ phase and completes it by completing the $1st$ phase, *i.e.*, the phase (s + 1) decays into phase s with rat $k_j\mu_j$. Now, using the above notations, the steady state equations can be written as:

$$\lambda\, p_0 = \sum_{i=1}^{m} k_i \mu_i\, {}_iP_{11} ,$$

$$(\lambda + k_j \mu_j)\, {}_jP_{1s} = k_j \mu_j\, {}_jP_{1,s+1} , \text{ for } 1 \leqslant s < k_j$$

$$(\lambda + k_j \mu_j)\, {}_jP_{1k_j} = \lambda_j p_0 + \frac{\lambda_j}{\lambda} \sum_{i=1}^{m} k_i \mu_i P_{21}$$

$$(\lambda + k_j \mu_j)\, {}_jP_{ns} = \lambda_j P_{n-1,s} + k_j \mu_j\, {}_jP_{n,s+1} , \text{ for } 1 \leqslant s < k_j , n \geqslant 2 ,$$

$$(\lambda + k_j \mu_j)\, {}_jP_{nk_j} = \lambda_j P_{n-1,K_j} + \frac{\lambda_j}{\lambda} \sum_{i=1}^{m} k_i \mu_i\, {}_iP_{n+1,1} \text{ for } n \geqslant 2.$$

The normalizing equation is

$$p_0 + \sum_{j=1}^{m} \sum_{n=1}^{\infty} \sum_{s=1}^{k_j} {}_jP_{ns} = 1. \tag{2.6}$$

To solve the above set of difference equations, the generating function technique, a device used by Baily[2], is used.

Define

$$H_j(y,z) = \sum_{n=1}^{\infty} \sum_{s=1}^{k_j} y^n z^s\, {}_jP_{ns}$$

and

$$H(y,z) = \sum_{j=1}^{m} H_j(y,z) \tag{2.8}$$

Then

$$H(1,1) = 1 - p_0 \tag{2.9}$$

Multiplying (2.1) – (2.5) by the appropriate powers of y and z and summing over n and s, we get

$$(\lambda + k_j \mu_j - \lambda y - \frac{k_j \mu_j}{z})\, H_j(y,z) = \lambda_j z^{k_j} (y-1)\, p_0$$

$$- k_j \mu_j G_j(y) + \frac{\lambda_j z^{k_j}}{\lambda y} \sum_{i=1}^{m} k_i \mu_j G_i(y)$$

where

$$\sum_{n=1}^{\infty} y^n \, {}_jp_{ni} \ , \ j = 1,2, \quad , m$$

Thus, we have an expression for $H_j(y,z)$ in terms of p_0 and $G_j(y)$. To find $G_j(y)$, we use one of the properties of the generating function. Since each one of the equations in (2.10) is valid for $|y| < 1$ and $|z| < 1$ then it is also valid when

$$Z = \frac{k_j \mu_j}{k_j \mu_j + \lambda(1-y)}$$

Substituting this value of z in (2.10) and

$$B_j(y) = [\frac{k_j \mu_j}{k_j \mu_j + \lambda(1-y)}]^{k_j}$$

we get

$$0 = \lambda_j (y-1) p_0 B_j - k_j \mu_j G_j + \frac{\lambda_j B_j}{\lambda y} \sum_{i=1}^{m} k_i \mu_i G_i \ , \ j = 1,2, \quad , m$$

which can be written as

$$\sum_{i=1}^{m} [k_j \mu_j \mu_{ji} - \frac{(\lambda_j B_j)(k_i \mu_i)}{\lambda y} \quad G_i = \lambda_j (y-1) p_0 B_j \ , \quad = \ ,2, \quad , m$$

where

$$\delta_{ji} \begin{cases} 1 & , \quad \text{for } j = i \ , \\ 0 & , \quad \text{for } j \neq i \ . \end{cases}$$

This set of equations can be rewritten in the matrix equation

$$\begin{bmatrix} G_1 \\ G_2 \\ \\ C_m \end{bmatrix} = (y-1) \ p_0 \begin{bmatrix} \lambda_1 \ B_1 \\ \lambda_2 \ B_2 \\ \\ \lambda_m \ B_m \end{bmatrix}$$

Where A is a square matrix with the elements

$$a_{ii} = k_i \mu_i - \frac{(\lambda_i B_i)(k_i \mu_i)}{\lambda y} \ , \ i = \ ,2, \quad , m \ ;$$

and

$$a_{ii} = \frac{(\lambda_i B_i)(k_j \mu_j)}{\lambda y} \ , \ i, j = \ ,2, \ ., m, \ i \neq j$$

Solving the matrix equation we get

$$\begin{bmatrix} G_1 \\ G_2 \\ \\ G_m \end{bmatrix} = (y - 1)\, p_0\, A^{-1} \begin{bmatrix} \lambda_1 B_1 \\ \lambda_2 B_2 \\ \cdots \\ \lambda_m B_m \end{bmatrix}$$

where the elements $\alpha_{ij}^{'s}$ of A^{-1} are:

$$\alpha_{ii} = \frac{\left(1 - \sum\limits_{j \neq i} \frac{\lambda_j B_j}{\lambda y}\right)}{k_i \mu_i \left(1 - \sum\limits_{j=1}^{m} \frac{\lambda_j B_j}{\lambda y}\right)}, \quad i = 1,2, \quad , m$$

and

$$\alpha_{ij} = \frac{\left(\frac{\lambda_j B_j}{\lambda y}\right.}{k_i \mu_i \left(1 - \sum\limits_{i=1}^{m} \frac{\lambda_j B_j}{\lambda y}\right)}, \quad i,j = \ ,2, \quad , m, \quad j \neq i.$$

we find that

$$G_i = \frac{\lambda_i (y-1)\, p_0\, B_i}{k_i \mu_i \left(1 - \sum\limits_{j=1}^{m} \frac{\lambda_j B_j}{\lambda y}\right.}$$

Substituting this result in (2.10), we get

$$H_j(y,z) = \frac{\lambda_j (1-y)\, p_0\, (B_j - Z^{k_j})}{[\lambda(1-y) + k_j \mu_j (1 - \frac{1}{z})]\left(1 - \sum\limits_{i=1}^{m} \frac{\lambda_i B_i}{\lambda y}\right)},$$

$$j = 1,2, \quad ., m$$

Now, it remains to obtain the unknown p_0.

It is observed that $H_j(1,z)$ attains the indeterminate form o/o. Then, applying L' Hospital's rule we find that

$$H_i(1,z) = \frac{\lambda_j p_0 \sum\limits_{s=1}^{k_j} z^{'}}{k_i \mu_i \left(1 - \sum\limits_{i=1}^{m} \frac{\lambda_i}{\mu_i}\right)}$$

from which we have

$$H_i\,(1,1) = \frac{\lambda_j\, p_0}{\mu_j\,(1 - \sum_{i=1}^{m} \frac{\lambda_i}{\mu_i})}$$

Now using equation (2.9), we can write

$$\frac{p_0 \sum_{j=1}^{m} \frac{\lambda_i}{\mu_i}}{1 - \sum_{i=1}^{m} \frac{\lambda_i}{\mu_i}} = - p_0 \,,$$

from which it is found that

$$p_0 = 1 - \sum_{i=1}^{m} \frac{\lambda_i}{\mu_i} \qquad (2.13)$$

Results and Discussion

The generating function $H_j(y,z)$ allows us to calculate what are called the measures of effectiveness of the system.

(a) The steady-state probability ${}_jp$, that the service facility is busy with a unit of type-j can be obtained as follows:

$${}_jp = \sum_{n=1}^{\infty} \sum_{s=1}^{k_j} {}_jp_{ns} = H_j\,(1,1) = \frac{\lambda_j}{\mu_i} \,,\; j = 1,2, \ldots, m \,.$$

(b) The expected number of units in the system, E(n), can be obtained as follows:

$$E(n) = \sum_{j=1}^{m} \sum_{n=1}^{\infty} \sum_{s=1}^{k_j} n\, {}_jp_{ns} = \frac{\partial H(y,z)}{\partial y}\Bigg|_{\substack{y=1\\z=1}} = \frac{d\, H\,(y,1)}{d\, y}\Bigg|_{y=1}$$

$$= \sum_{i=1}^{m} \frac{\lambda_i}{\mu_i} + \frac{\lambda^2 \sum_{i=1}^{m} \frac{\lambda_i\,(k_i + 1)}{\lambda\, k_i\, \mu_i^2}}{2\,(1 - \sum_{i=1}^{m} \frac{\lambda_i}{\mu_i})} \,,$$

which agrees with the Pollaczek-Khinchine formula[3] for the system M/G/1.

(c) The expected number of units in the queue, E(n – 1) can be obtained as follows:

$$E(n-1) = \sum_{j=1}^{m} \sum_{n=1}^{\infty} \sum_{s=1}^{k_j} (n-1)\, {}_jP_{ns} = E(n) - (1 - p_0)$$

$$E(n) - \sum_{i=1}^{m} \frac{\lambda_i}{\mu_i}$$

$$= \frac{\lambda^2 \sum_{i=1}^{m} \frac{\lambda_i (k_i + 1)}{\lambda k_i \mu_i^2}}{2 \left(1 - \sum_{i=1}^{m} \frac{\lambda_i}{\mu_i}\right)}$$

(d) Next, we drive the expected waiting time of a unit in the queue, $E(w_q)$. In the description of the system, we assumed that the units are served in order of their arrival. If a unit arrives when the system is in state (n,j,s), the expected waiting time in the queue of that particular unit[4,5] is:

$$(n-1) \sum_{i=1}^{m} \frac{\lambda_i}{\lambda} \cdot \frac{1}{\mu_i} + \frac{s}{k_j \mu_j}$$

Then, the expected waiting time in the queue of any unit is

$$E(w_q) = \sum_{i=1}^{m} \sum_{n=1}^{\infty} \sum_{s=1}^{k_j} \left[\sum_{i=1}^{m} \frac{(n-1)\lambda_i}{\lambda \mu_i} + \frac{s}{k_j \mu_j} \right] {}_jP_{ns}$$

$$= E(n-1) \sum_{i=1}^{m} \frac{\lambda_i}{\lambda \mu_i} + \sum_{j=1}^{m} \frac{1}{k_j \mu_j} \cdot \left. \frac{d H_j(1,z)}{dz} \right|_{z=1}$$

Substituting for $E(n-1)$ and using (2.12), we can write

$$E(w_q) = \frac{\lambda \sum_{i=1}^{m} \frac{\lambda_i (k_i + 1)}{\lambda k_i \mu_i^2}}{2 \left(1 - \sum_{i=1}^{m} \frac{\lambda_i}{\mu_i}\right)}$$

(e) We can also obtain the expected waiting time in the system of type-j, which we denote by $E_j(w)$, as

$$E_j(w) = E(w_q) + \frac{1}{\mu_i}$$

$$\frac{1}{\mu_i} + \frac{\lambda \sum_{i=1}^{m} \frac{\lambda_i (k_i + 1)}{\lambda k_i \mu_i^2}}{2 \left(1 - \sum_{i=1}^{m} \frac{\lambda_i}{\mu_i}\right)}$$

(f) We finally obtain the expected waiting time in the system of a unit, E(w) as

$$E(w) = \sum_{i=1}^{m} \frac{\lambda_i}{\lambda} \cdot \frac{1}{\mu_i} + \frac{\lambda \sum_{i=1}^{m} \frac{\lambda_i (k_i + 1)}{\lambda k_i \mu_i^2}}{2\left(1 - \sum_{i=1}^{m} \frac{\lambda_i}{\mu_i}\right)}$$

Acknowledgement

I would like to thank the referees for their helpful comments.

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الصفوف ذات المدخلات البواسونية المتعددة وفترات الخدمة التي تتبع توزيع إرلانج

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وبفرض أن جميع الوحدات التي تصل إلى النظام تبقى فيه حتى الانتهاء من خدمتها فإننا نحصل على الدوال المولدة للاحتمالات والمقاييس الخاصة بهذا النظام .